Long-Term Interest Rates and Consol Bond Valuation

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Abstract

This paper presents a Gaussian three-factor model of the term structure of interest rates which is Markov and time-homogeneous. The model captures the whole term structure and is particularly useful in forward simulations for applications in long-term swap and bond pricing, risk management and portfolio optimization. Kalman filter parameter estimation uses EU swap rate data and is described in detail. The yield curve model is fitted to data up to 2002 and assessed by simulation of yield curve scenarios over the next two years. It is then applied to the valuation of callable floating rate consol bonds as recently issued by European banks over the subsequent period 2005 to 2007.

JEL Classification: G10, G12.

Keywords: Multifactor Term Structure Model; Kalman Filter; Simulation, Consol Bonds

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1. Introduction

The literature in the area of interest rate modelling is extensive. Traditional term structure models, such as Vasicek (1977) and Cox, Ingersoll and Ross (CIR, 1985) specify the short rate process. As short-term and long-term rates are not perfectly correlated, the data are clearly inconsistent with the use of one-factor time-homogeneous models. Chan *et al.* (1992) demonstrate the empirical difficulties of one-factor continuous-time specifications within the Vasicek and CIR class of models using the generalized methods of moments.

Litterman and Scheinkman (1991) find that 96% of the variability of the returns of any risk free zero-coupon bond can be explained by three factors: the *level*, *steepness* and *curvature* of the yield curve. They also point out that the 'correct model' of the term structure may involve unobservable factors. For instance, it is widely believed that changes in the Federal Reserve policy are a major source of changes in the shape of the US yield curve. Even though the Federal Reserve policy is itself observable, it is not clear how to measure its effect on the yield curve. Litterman and Scheinkman (1991) themselves used unobservable factors in their approach by applying principal component analysis.

Most term structure models such as Ho and Lee (1986), Hull and White (1990) and Heath, Jarrow and Morton (1989) are specified using the risk-neutral measure corresponding to a complete market. This makes them appropriate for relative-pricing applications, but inappropriate for forward simulations, which needs to take place under the market measure in an incomplete market. An exception is Rebonato *et al.* (2005) who focus on yield curve evolution under the market measure and present a semi-parametric method to explain the yield curve evolution. Ho and Lee (1986) and Heath, Jarrow and Morton (1992) introduced a new approach to interest rate modelling in which they fit the initial term structure exactly. Duffie and Kan (1996) developed a general theory for multifactor affine versions of these models with coefficients obtained analytically. The book by James and Webber (2000) gives a comprehensive summary of development to 2000. See also Brigo and Mercurio (2007) and Wu (2009) for a more recent summaries.

In spite of significant theoretical achievements there are still difficulties with the long-term forecasting of future yields. The affine arbitrage-free version of the Nelson-Siegel (1987) model by Christensen *et al.* (2007) investigates the gap between the theoretically rigorous risk-neutral models used for pricing and the empirical tractability required by econometricians for forecasting and offers some improvements in forecasting performance.

In our research we focus on the development of a model that allows simulation of long-term scenarios for the yield curve which include the market prices of risk in the dynamics (Medova *et al*., 2006). Historically low interest rates in recent years have emphasized the importance of accurately valuing long-term guarantees (Wilkie *et al.*, 2004, Dempster *et al*., 2006). The asset liability management of funds behind the guaranteed return products offered by pension providers must be based on yield curve modelling (Dempster *et al*., 2007, 2009). Banks also face new pricing challenges due to the increased demand for long-maturity derivatives to hedge insurance and pension liabilities (for liability driven investment) and therefore require good long-term interest rate models.

Figure 1 shows the development over time of short- and long-term interest rates in the Eurozone for the period 1997-2002. Figure 2 plots the weekly standard deviations of the yields over the same period.

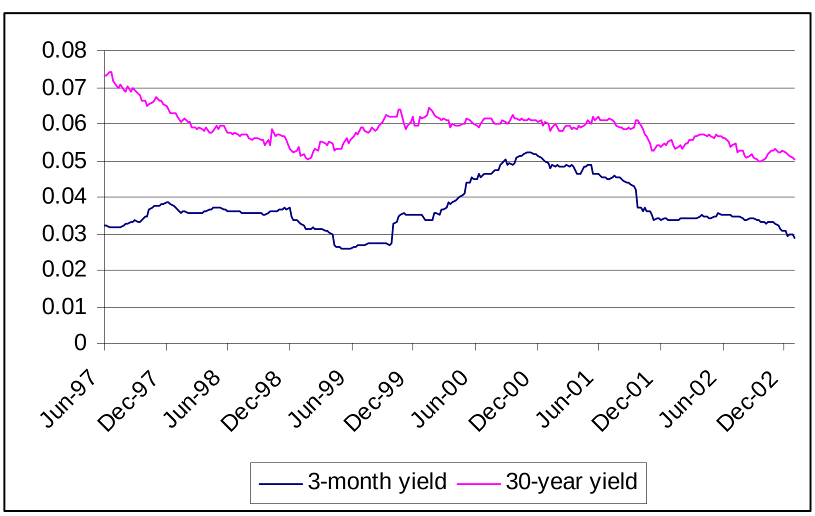

Figure 1. 3-month and 30-year EU yields for the period June 1997 to December 2002

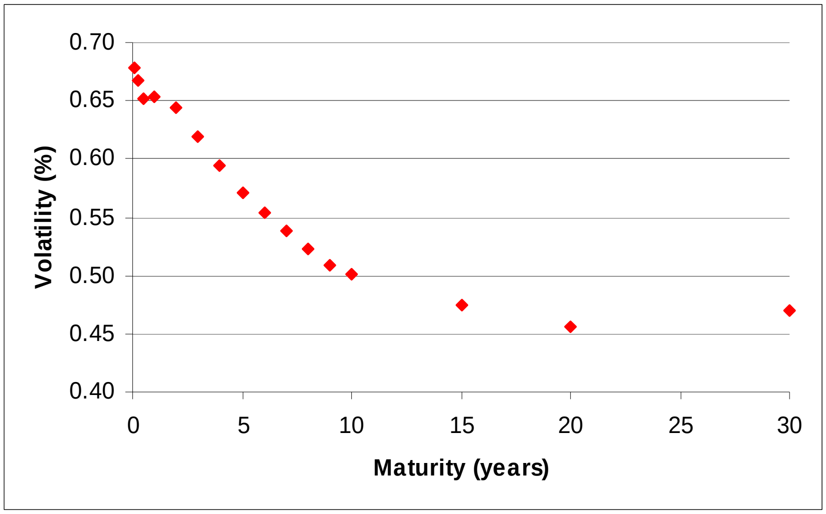

Figure 2. Weekly standard deviation of yields for the period June 1997 to Dec 2002

In this paper we focus on a term structure model with the following characteristics:

- The model is set in a continuous-time framework. This allows implementation in discrete time with any length of time step Δt without the need to construct a new model each time we change Δt. This is an important requirement for the flexibility of forward simulations.
- Interest-rate dynamics are consistent with what we observe in historical data.
- The affine class model has a closed-form solution for bond pricing, permitting straightforward analytical calculation of bond prices in forward simulation.
- The short rate is mean-reverting.
- The model permits a tractable method of estimation and calibration.
- The model is flexible enough to give rise to a range of different yield curve shapes and dynamics (steepening, flattening, yield curve inversion, etc.).

The remainder of the paper is structured as follows. In Section 2, the three-factor Gaussian term structure model is introduced and a closed-form solution for bond prices derived. Section 3 discusses the state-space formulation of the model and the estimation of its parameters using the Kalman filter and numerical likelihood maximization. The data and empirical analyses, focussing on fitting the data as well as on the simulation potential of the model, are presented in Section 4. Section 5 applies the 3-factor model to the pricing of a representative consol bond and Section 6 concludes.

2. Three-Factor Term Structure Model

The term structure model presented in this paper is driven by three factors and can be viewed as an extension to the generalized Vasicek model of Langetieg (1980) which includes a third factor. The first two factors[1] **X** and **Y** satisfy the standard Vasicek stochastic differential equations with *mean reversion rates* λ_X and λ_Y and *levels* $\frac{\mu_X}{\lambda_X}$ and $\frac{\mu_Y}{\lambda_Y}$ respectively. The innovation of our model is in the treatment of the instantaneous short rate **R**. The mean reversion of **R** with rate k has a level that is *stochastic* rather than deterministic and depends on the level of the other two factors **X** and **Y** driving the model. The **X** and **Y** factors may be interpreted respectively as a long rate and (minus) the slope of the yield curve from a perceived (instantaneous) short rate $\mathbf{R}^* := \mathbf{X} + \mathbf{Y}$.

Risk neutral measure

Starting from the formulation of the model under the risk neutral measure Q we have the following three stochastic differential equations (SDEs) for the factors

$$d\mathbf{X}_t = (\mu_X - \lambda_X X_t)dt + \sigma_X d\tilde{\mathbf{W}}_t^X \tag{1}$$

$$d\mathbf{Y}_t = (\mu_Y - \lambda_Y Y_t)dt + \sigma_Y d\tilde{\mathbf{W}}_t^Y \tag{2}$$

$$d\mathbf{R}_t = k(\boldsymbol{X}_t + \mathbf{Y}_t - R_t)dt + \sigma_R d\tilde{\mathbf{W}}_t^R, \tag{3}$$

where the $d\tilde{\mathbf{W}}$ terms are correlated. Factoring the covariance matrix of the $d\tilde{\mathbf{W}}$ terms using a Cholesky decomposition into the product of a transposed upper and a lower triangular square root matrix results in a new formulation of the form

$$d\mathbf{X}_t = (\mu_X - \lambda_X X_t)dt + \sum_{i=1}^{3} \sigma_{X_i} d\mathbf{Z}_t^i \tag{4}$$

$$d\mathbf{Y}_t = (\mu_Y - \lambda_Y Y_t)dt + \sum_{i=1}^{3} \sigma_{Y_i} d\mathbf{Z}_t^i \tag{5}$$

[1] We use boldface throughout the paper to denote random or conditionally random entities.

$$d\mathbf{R}_t = k(X_t + Y_t - R_t)dt + \sum_{i=1}^{3} \sigma_{R_i} d\mathbf{Z}_t^i, \tag{6}$$

where the $d\mathbf{Z}$ terms are uncorrelated.

Closed form solution

The solution follows the usual steps. We first solve the SDEs for **X, Y** and **R** to obtain the price of a zero-coupon bond at time *t* paying 1 at time *T*

$$P(t,T) = \mathbb{E}_t^Q \left\{ \exp\left(-\int_t^T \mathbf{R}_s ds \right) \right\}, \tag{7}$$

where $\mathbb{E}_t^Q$ denotes the expectation under the risk neutral measure *Q* conditional on the information at time *t*. As $\mathbf{R}_s$ is normally distributed in our model we can use the moment generating function for the normal distribution to rewrite (7) as

$$P(t,T) = \exp\left\{ \mathbb{E}_t^Q \left(-\int_t^T \mathbf{R}_s ds \right) + \frac{1}{2} \text{var}_t^Q \left(-\int_t^T \mathbf{R}_s ds \right) \right\}, \tag{8}$$

where var_t^Q denotes the conditional variance under *Q*. Integrating the solution of the SDE for **R** and taking the expectation and variance of the result gives expressions for the two terms in (8) involving (to simplify notation) the parameters

$$\begin{aligned} m_{X_i} &= -\frac{k\sigma_{X_i}}{\lambda_X (k - \lambda_X)} \\ m_{Y_i} &= -\frac{k\sigma_{Y_i}}{\lambda_Y (k - \lambda_Y)} \\ n_i &= \frac{\sigma_{X_i}}{k - \lambda_X} + \frac{\sigma_{Y_i}}{k - \lambda_Y} - \frac{\sigma_{R_i}}{k} \\ p_i &= -(m_{X_i} + m_{Y_i} + n_i). \end{aligned} \tag{9}$$

Hence (see e.g. Medova *et al*., 2006)

$$P(t,T) = e^{-y_{t,T}(T-t)} = \exp\left\{ -A(t,T)R_t - B(t,T)X_t - C(t,T)Y_t - D(t,T) \right\} \tag{10}$$

with corresponding yield to maturity

$$y_{t,T} = \frac{A(t,T)R_t + B(t,T)X_t + C(t,T)Y_t + D(t,T)}{T-t}, \tag{11}$$

where

$$A(t,T) = \frac{1}{k}(1 - e^{-k(T-t)}) \tag{12}$$

$$B(t,T)=\frac{k}{k-\lambda_X}\left\{\frac{1}{\lambda_X}(1-e^{-\lambda_X(T-t)})-\frac{1}{k}(1-e^{-k(T-t)})\right\} \tag{13}$$

$$C(t,T)=\frac{k}{k-\lambda_Y}\left\{\frac{1}{\lambda_Y}(1-e^{-\lambda_Y(T-t)})-\frac{1}{k}(1-e^{-k(T-t)})\right\} \tag{14}$$

$$\begin{aligned}
D(t,T)=&\left(T-t-\frac{1}{k}(1-e^{-kT})\right)\left(\frac{\mu_X}{\lambda_X}+\frac{\mu_Y}{\lambda_Y}\right)-\frac{\mu_X}{\lambda_X}B(t,T)-\frac{\mu_Y}{\lambda_Y}C(t,T)\\
&-\frac{1}{2}\sum_{i=1}^{3}\left\{\frac{m_{X_i}^2}{2\lambda_X}(1-e^{-2\lambda_X(T-t)})+\frac{m_{Y_i}^2}{2\lambda_Y}(1-e^{-2\lambda_Y(T-t)})\right.\\
&+\frac{n_i^2}{2k}(1-e^{-2k(T-t)})+p_i^2(T-t)+\frac{2m_{X_i}m_{Y_i}}{\lambda_X+\lambda_Y}(1-e^{-(\lambda_X+\lambda_Y)(T-t)})\\
&+\frac{2m_{X_i}n_i}{\lambda_X+k}(1-e^{-(\lambda_X+k)(T-t)})+\frac{2m_{X_i}p_i}{\lambda_X}(1-e^{-\lambda_X(T-t)})\\
&+\frac{2m_{Y_i}n_i}{\lambda_Y+k}(1-e^{-(\lambda_Y+k)(T-t)})+\frac{2m_{Y_i}p_i}{\lambda_Y}(1-e^{-\lambda_Y(T-t)})\\
&\left.+\frac{2n_ip_i}{k}(1-e^{-k(T-t)})\right\}.
\end{aligned} \tag{15}$$

Market measure

Bond pricing is achieved under the *risk-neutral* measure Q. However, for the model to be used for forward simulations, we need to adjust the set of stochastic differential equations so that we capture the model dynamics under the *market* (or real-world) measure P by adding a *risk premium* to each drift term. The risk premium is given by the *market price of risk* γ times the quantity of risk and it is generally assumed in a Gaussian specification that the quantity of risk is given by the *volatility* of each factor. We assume that the market prices of risk are independent of the time to maturity of the bond and are not functionally dependent on the factor being modelled.

The set of processes under the market measure thus satisfy

$$d\mathbf{X}_t=(\mu_X-\lambda_X X_t+\gamma_X\sigma_X)dt+\sigma_X d\tilde{\mathbf{W}}_t^X \tag{16}$$

$$d\mathbf{Y}_t=(\mu_Y-\lambda_Y Y_t+\gamma_Y\sigma_Y)dt+\sigma_Y d\tilde{\mathbf{W}}_t^Y \tag{17}$$

$$d\mathbf{R}_t=\{k(X_t+Y_t-R_t)+\gamma_R\sigma_R\}dt+\sigma_R d\tilde{\mathbf{W}}_t^R, \tag{18}$$

where all three factors contain a market price of risk γ in volatility units.

3. Estimation Procedure

The estimation of affine term structure models is known to be problematic due to the existence of numerous model likelihood maxima with essentially identical fit to the data (Kim and Orphanides, 2005). Babbs and Nowman (1999) applied the Kalman filter to estimate the two-factor generalised Vasicek model. Some other examples of the literature on filtering methods are Chen and Scott (1993), De Jong (2000), De Jong and Santa-Clara (1999), Geyer and Pichler (1997) and Duffee (2002). Most of these papers analyze multi-factor versions of the Cox-Ingersoll-Ross (CIR) model using mutually independent factors. De Jong (2000) extends this approach to the more general class of affine models proposed by Duffie and Kan (1996).

Kalman filter

Here we describe in detail the Kalman filter estimation procedure (Harvey,1989), for our three factor yield curve model in state-space form, which simultaneously integrates time-series and cross-sectional aspects of the model. It also allows the identification of the market prices of interest rate risk critical for forward simulation.

The general state-space form applies to multivariate time series. The N observable variables $\mathbf{y}_t$ at time t (here zero coupon bond yields of various maturities) are related to a vector $\boldsymbol{\alpha}_t$ known as the *state vector* (here our three yield curve factors) via a *measurement equation*

$$\mathbf{y}_t = \mathrm{Z}\alpha_t + \mathrm{d} + \boldsymbol{\varepsilon}_t \qquad t = 1, \ldots, T, \tag{19}$$

where $\mathbf{Z}$ is an $N \times m$ matrix, α_t is an mx1 vector, d and $\boldsymbol{\varepsilon}_t$ are $N \times 1$ vectors and the error term is assumed to consist of serially uncorrelated disturbances with mean zero and covariance matrix H, i.e.

$$\mathbb{E}(\boldsymbol{\varepsilon}_t) = 0 \qquad \mathrm{var}(\boldsymbol{\varepsilon}_t) = \mathrm{H}. \tag{20}$$

In general Z, d and H may depend on t.

Even though the elements of the state $\boldsymbol{\alpha}_t$ are unobservable, they are known to follow a first-order Markov process specified by the *transition equation*

$$\boldsymbol{\alpha}_t = \mathrm{A}\alpha_{t-1} + \mathrm{c} + \mathrm{S}\boldsymbol{\eta}_t \qquad t = 1, \ldots, T, \tag{21}$$

where A is an $m \times m$ matrix, c an $m \times 1$ vector, S an $m \times g$ matrix and $\boldsymbol{\eta}_t$ a $g \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix Q, that is

$$\mathbb{E}(\boldsymbol{\eta}_t) = 0 \qquad \text{var}(\boldsymbol{\eta}_t) = \text{Q}. \tag{22}$$

Again, in general T, c and S may depend on t, however we will treat here the *time-homogeneous* case appropriate to the long term[2].

Two further assumptions will be required to complete the state-space formulation:

- The initial state vector $\boldsymbol{\alpha}_0$ has mean a_0 and covariance matrix P_0, that is

$$\mathbb{E}(\boldsymbol{\alpha}_0) = \text{a}_0 \qquad \text{var}(\boldsymbol{\alpha}_0) = \text{P}_0. \tag{23}$$

- The disturbance terms $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are uncorrelated with each other in all time periods and uncorrelated with the initial state, that is

$$\mathbb{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\eta}_s') = 0 \qquad \text{for all } s, t = 1, \ldots, T \tag{24}$$

and

$$\mathbb{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\alpha}_0') = 0 \qquad \mathbb{E}(\boldsymbol{\eta}_t \boldsymbol{\alpha}_0') = 0 \qquad t = 1, \ldots, T. \tag{25}$$

The important concept behind the state space formulation is this separation of the noise driving the system dynamics $\boldsymbol{\eta}_t$ and the observational noise $\boldsymbol{\varepsilon}_t$.

The *Kalman filter* is applied recursively in order to compute the optimal estimator of the state vector at time t given all the information currently available, which consists of the observations up to and including y_t. Assuming a Gaussian state space, the disturbances and the initial state vector will be normally distributed.

In a state-space model the system matrices depend on a set of unknown parameters (in our case 14) referred to as *hyper-parameters* and defined in Table 1 below. Using the Kalman filter to construct the likelihood function and then maximizing it using a suitable numerical optimization procedure, we can carry out maximum likelihood estimation of the hyper-parameters. The joint probability of a set of T observations

[2] This assumption implies that the conditional variance of yield changes is constant over time. A number of studies concerned with the relatively short term have found that yield changes are conditionally heteroskedastic, *cf.* Ball and Torous (1999). Fong and Vasicek (1991) introduced stochastic volatility to represent this situation, whose relevance to the long run is questionable, for pricing (see also Litterman *et al.*, 1991 and Andersen *et al.*, 2004).

can be expressed in terms of conditional distributions. For a multivariate normal distribution we have

$$L(\mathbf{y};\boldsymbol{\varphi}) = \prod_{t=1}^{T} p(\mathbf{y}_t \,|\mathbf{Y}_{t-1}), \tag{26}$$

where $p(\mathrm{y}_t \mid \mathrm{Y}_{t-1})$ is the distribution of $\mathbf{y}_t$ conditional on the information at time $t-1$, i.e. $\mathrm{Y}_{t-1} = (y_{t-1}, y_{t-2}, ..., y_1)'$. Since we have a Gaussian model we can write the log-likelihood function in *prediction error decomposition form* as

$$\log L(\varphi) = -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left|\mathrm{F}_t\right| - \frac{1}{2}\sum_{t=1}^{T}\mathrm{v}_t'\mathrm{F}_t^{-1}\mathrm{v}_t, \tag{27}$$

where F_t is estimated by the covariance matrix obtained from the Kalman filter as

$$\mathrm{F}_t = \mathrm{Z}\mathrm{P}_{t|t-1}\mathrm{Z}' + \mathrm{H} \tag{28}$$

and v_t is the vector of *prediction errors* given by

$$\mathrm{v}_t = \mathrm{y}_t - \tilde{\mathrm{y}}_{t|t-1} = \mathrm{Z}(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t. \tag{29}$$

Together with the following two equations, (28) and (29) form the *measurement update equations*

$$\mathrm{a}_t = \mathrm{a}_{t|t-1} + \mathrm{P}_{t|t-1}\mathrm{Z}'\mathrm{F}_t^{-1}\mathrm{v}_t \tag{30}$$

$$\mathrm{P}_t = \mathrm{P}_{t|t-1} - \mathrm{P}_{t|t-1}\mathrm{Z}'\mathrm{F}_t^{-1}\mathrm{P}_{t|t-1}. \tag{31}$$

So first we specify starting values for the parameters. With these starting values we run the Kalman filter to obtain estimated yields and a time series for the unobserved state variables. Next, the parameters are estimated by maximizing the log-likelihood using the state variable path estimates as observations. The optimized parameter values are then used as the starting values for the next iteration of the Kalman filter. This loop continues until we obtain the optimal parameter estimates by this generalized EM algorithm (Dempster *et al.*, 1977). The calibration code is implemented in C++ and the optimization is performed using a combination of global (Direct, see Jones *et al.*, 1993) and local (approximate) conjugate direction (Powell, 1964) or derivative-free quasi-Newton (NAG BFGS used in Section 5) numerical algorithms.

The starting values for the Kalman filter are given by the mean and the covariance of the unconditional distribution of the stationary state vector. The state vector is stationary if c and A are time invariant and $\left|\lambda(\mathrm{A})\right| < 1$, where $\lambda(\mathrm{A})$ is the leading eigenvalue of A. In this case the mean a_0 is given by the unique solution to

$$\mathrm{a}_0 = \mathrm{A}\mathrm{a}_0 + \mathrm{c} \qquad \text{given by} \qquad \mathrm{a}_0 = (\mathrm{I} - \mathrm{A})^{-1}\mathrm{c} \tag{32}$$

and the covariance matrix P_0 will be given by the unique solution to the *Riccati equation*

$$\mathrm{P}_0 = \mathrm{A}\mathrm{P}_0\mathrm{A}' + \mathrm{SQS}' \quad \text{given by} \quad vec(\mathrm{P}_0) = (\mathrm{I} - \mathrm{A} \otimes \mathrm{A})^{-1} vec(\mathrm{SQS}'). \tag{33}$$

State space form

In our case the observable variables are given by risk free (Treasury) yields of different maturities, and are related to the vector of unobservable state variables $(\mathbf{X}, \mathbf{Y}, \mathbf{R})$ via the measurement equation. The measurement equation is obtained using (11) and adding serially and cross-sectionally uncorrelated disturbances with mean zero to take into account non-simultaneity of the observations, errors in the data, etc. The unobservable state variables are generated via the transition equations, which in our case are given by the discretized versions of (1), (2) and (3), using Euler's first order approximation[3], i.e.

$$\mathbf{X}_{t+\Delta t} = X_t + (\mu_X - \lambda_X X_t + \gamma_X \sigma_X)\Delta t + \sigma_X \sqrt{\Delta t}\,\boldsymbol{\eta}_{t,X} \tag{34}$$

$$\mathbf{Y}_{t+\Delta t} = Y_t + (\mu_Y - \lambda_Y Y_t + \gamma_Y \sigma_Y)\Delta t + \sigma_Y \sqrt{\Delta t}\,\boldsymbol{\eta}_{t,Y} \tag{35}$$

$$\mathbf{R}_{t+\Delta t} = R_t + (k(X_t + Y_t - R_t) + \gamma_R \sigma_R)\Delta t + \sigma_R \sqrt{\Delta t}\,\boldsymbol{\eta}_{t,R}. \tag{36}$$

In matrix form the transition equations can be written as

$$\begin{pmatrix} \mathbf{X}_t \\ \mathbf{Y}_t \\ \mathbf{R}_t \end{pmatrix} = \mathrm{A} \begin{pmatrix} X_{t-\Delta t} \\ Y_{t-\Delta t} \\ R_{t-\Delta t} \end{pmatrix} + \mathrm{c} + \mathrm{S}\boldsymbol{\eta}_t, \tag{37}$$

where

$$\mathrm{A} := \begin{pmatrix} 1 - \lambda_X \Delta t & 0 & 0 \\ 0 & 1 - \lambda_Y \Delta t & 0 \\ k\Delta t & k\Delta t & 1 - k\Delta t \end{pmatrix} \tag{38}$$

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[3] Alternatively, De Jong (2000) presents a general way to obtain the exact discrete-time state distributions in affine class models. As the benefits are unclear for our purposes and simulation complexity increases, we have not pursued this approach here.

$$c = \begin{pmatrix} (\mu_X + \gamma_X \sigma_X)\Delta t \\ (\mu_Y + \gamma_Y \sigma_Y)\Delta t \\ \gamma_R \sigma_R \Delta t \end{pmatrix} \tag{39}$$

$$S = \begin{pmatrix} \sigma_X \sqrt{\Delta t} & 0 & 0 \\ 0 & \sigma_Y \sqrt{\Delta t} & 0 \\ 0 & 0 & \sigma_R \sqrt{\Delta t} \end{pmatrix} \tag{40}$$

and $\boldsymbol{\eta}_t$ is a vector with serially uncorrelated disturbances satisfying

$$\mathbb{E}(\boldsymbol{\eta}_t) = 0 \qquad \operatorname{var}(\boldsymbol{\eta}_t) = \begin{pmatrix} 1 & \rho_{XY} & \rho_{XR} \\ \rho_{XY} & 1 & \rho_{YR} \\ \rho_{XR} & \rho_{YR} & 1 \end{pmatrix}. \tag{41}$$

In the current literature, several approaches have been adopted to estimate the covariance matrix of the measurement errors. For example, De Jong and Santa Clara (1999) used a spherical covariance matrix, $H = hI$, whereas Babbs and Nowman (1999) use a diagonal matrix. De Jong (2000) uses a full covariance matrix. We adopt a diagonal covariance matrix approach, optimizing likelihood using one-at-a-time search with the parameters divided into two groups: in the first search the model parameters are optimized followed by the minimization of the measurement errors in the second search. This process is repeated until convergence. The one-at-a-time search method is preferred over the full optimization with 14 model parameters and 16 measurement errors due to the scale of the optimization problem in the combined case. Even though the full covariance matrix is to be highly preferred, we have avoided this specification since using yields of 16 different maturities would result in 136 noise parameters to be estimated.

Estimation results

For our empirical analysis yields on ordinary (par) fixed-for-floating rate Euro swap contracts[4] are used as data. Since the swap market is highly liquid with many par swaps traded every day, it is possible to obtain rates for a set of swaps with *constant*

[4] A *par interest rate swap* is a standard contract between two counterparties to exchange cash flows. At set time intervals termed *reset dates* one pays a predetermined *fixed* rate of interest on the *nominal* value, the other a *floating* rate, until the *maturity* date of the contract. The floating leg of swap fixes the interest rates for each payment at the rate of a published interest rate. The fixed rate, known as the *swap rate,* is that interest rate which makes the fair value of the par swap 0 at inception. Thus the cash flows of the two legs of a par swap are those of a pair of bonds with face value the swap nominal, one fixed rate, and the other floating rate.

maturities from 1 to 30 years from the market. From the market swap rates a swap curve which gives the rates for constant maturity swaps (CMS) of *all* durations may be constructed each day. Dai and Singleton (2000) point out that these yields are preferable for analysis for the following reasons. The swap markets provide 'constant maturity' yield data, whereas in the Treasury market the maturities of 'constant maturity' yields are only approximately constant or the data represent interpolated series. Additionally, the on-the-run (i.e. just purchased at auction) treasuries that are often used in empirical studies are typically on 'special' (haircut) in the repo market to which they are immediately (albeit temporarily) sold. So, strictly speaking, the Treasury data should be adjusted for repo specials prior to any empirical analysis. Unfortunately, the requisite data for making these adjustments are not readily available, and consequently such adjustments are rarely made.

For estimation and calibration purposes, we first use weekly 1, 3 and 6 month EU LIBOR and Euro swap data for the period June 1997 to December 2002 (a total of 292 time points) for 16 different yields with maturities equal to 1, 3 and 6 months and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20 and 30 years. The sample period was determined on one end by the unavailability of reliable long-term swap data for years prior to 1997 and on the other end by its use for backtests over the difficult 2003-2004 period in the bond markets[5]. We interpolate the swap curve linearly to obtain swap rates at all maturities and then use the data recursively from the 1 month rate to back out a zero-coupon bond yield curve from the basic swap pricing equation for each week[6]. This derived data is the input for model calibration. The estimation results are presented in Table 1 all have plausible values. Bearing in mind that the factor **y** is a *negative* yield curve slope between a market expected short rate and the long rate, all the signs in Table 1 are as expected. All parameter estimates are statistically significant at the 1% level, unlike the estimates found by Babbs and Nowman (1999), who looked at Kalman filtering generalized Vasicek models. However they only used yields of eight different maturities and Geyer and Pichler (1999) show that a larger number of maturities is important to improve the precision of the parameter estimates. Shocks to

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[5] But see Section 5 where more recent data up to 2008 is used.

[6] We also evaluated quadratic interpolation but deemed the negligible improvement in accuracy not worth the considerable increase in computational burden.

the long rate and the expected yield curve slope decay here with half lives over 4 years and about 6 months respectively.

Euro Data		Estimated Value	Standard Error
Long term risk neutral mean *X*	μ_X / λ_X	0.199	1.69E-04
Long term risk neutral mean *Y*	μ_Y / λ_Y	-0.134	1.69E-04
Speed of mean reversion *X*	λ_X	0.161	1.03E-03
Speed of mean reversion *Y*	λ_Y	1.332	6.87E-03
Speed of mean reversion *R*	k	0.117	1.64E-03
Volatility *X*	σ_X	0.030	1.89E-04
Volatility *Y*	σ_Y	0.186	9.80E-04
Volatility *R*	σ_R	0.006	2.26E-04
Correlation *X* and *Y*	ρ_{XY}	-0.642	6.94E-03
Correlation *X* and *R*	ρ_{XR}	0.177	1.82E-02
Correlation *Y* and *R*	ρ_{YR}	-0.540	1.81E-02
Market price of risk for *X*	γ_X	0.556	3.91E-03
Market price of risk for *Y*	γ_Y	-1.017	5.50E-03
Market price of risk for *R*	γ_R	0.096	1.65E-02

Table 1. Estimated parameters using the Kalman filter

Table 2 provides the estimated standard deviations $\sqrt{h_i}$ of the measurement errors, where h_i is the ith diagonal element of the covariance matrix H. In particular, these standard deviations range from less than 1 basis point for the seven-year yield to 24 basis points for the thirty-year rate. These measurement errors are all significant at the 1% level and compare in magnitude to those in Babbs and Nowman (1999) and very favourably to studies by, for example Chen and Scott (1993) and Geyer and Pichler (1996), who both estimate the multifactor Cox-Ingersoll-Ross model on U.S. data.

	Maturity	Estimated Value	Standard Error
$\sqrt{h_1}$	1 month	1.57E-03	6.63E-05
$\sqrt{h_2}$	3 months	8.64E-04	3.81E-05
$\sqrt{h_3}$	6 months	1.55E-04	3.19E-05
$\sqrt{h_4}$	1 year	6.71E-04	2.96E-05
$\sqrt{h_5}$	2 years	5.08E-04	2.15E-05
$\sqrt{h_6}$	3 years	2.85E-04	1.21E-05
$\sqrt{h_7}$	4 years	1.49E-04	7.03E-06
$\sqrt{h_8}$	5 years	4.96E-05	4.59E-06
$\sqrt{h_9}$	6 years	6.58E-05	2.89E-06
$\sqrt{h_{10}}$	7 years	1.00E-05	3.83E-06
$\sqrt{h_{11}}$	8 years	9.44E-05	4.1E-06
$\sqrt{h_{12}}$	9 years	1.75E-04	7.63E-06
$\sqrt{h_{13}}$	10 years	2.94E-04	1.28E-05
$\sqrt{h_{14}}$	15 years	7.45E-04	3.14E-05
$\sqrt{h_{15}}$	20 years	1.23E-03	5.32E-05
$\sqrt{h_{16}}$	30 years	2.37E-03	1.03E-04

Table 2. Measurement errors

A general limitation of affine yield curve models with mean reversion is that the volatility of long rates tends to decay too rapidly with maturity relative to that exhibited by market data. Our use of maturity specific volatilities for the measurement errors compensates for this effect. Indeed, the standard deviation of the measurement errors for the 15, 20 and 30 years rates shown in Table 2 are significantly greater than those for the other maturities. More generally, and similar to Geyer and Pichler (1996), the error standard deviations exhibit a distinct U-shaped pattern as depicted in Figure 3. A possible explanation for this might be that the observed data for medium range maturities are highly correlated and therefore easier to fit. The short rate behavior in Figure 3 also indicates that the use of the one-month yield as a proxy for the instantaneous short rate is likely to give rise to problems. In general, 1 month and

6 month Libor rate measurement errors appear inconsistent with those for rates derived from the swap data for maturities in years, probably due to liquidity factors.

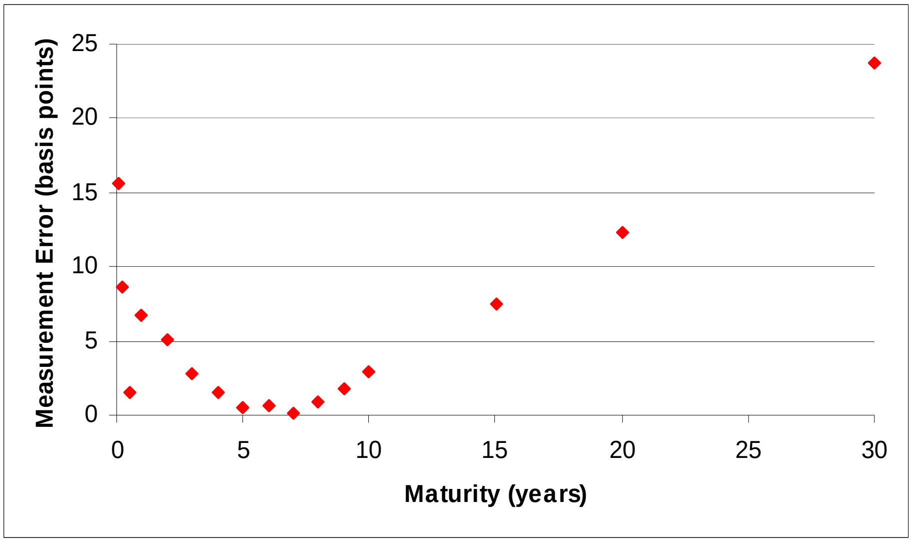

Figure 3. Measurement error of the fitted yields

Factor loadings

Like Babbs and Nowman (1999), we also look at the factor loadings of our three-factor model as a function of maturity to determine the nature of the factors calculated by the Kalman filter. The resulting curve for each factor represents the change in yield caused by a shock to that factor of one standard deviation magnitude so that all shocks are equally likely events (Litterman and Scheinkman, 1991). As factor loadings correspond to orthogonal Brownian motions, rather than those with correlated innovations, we first use Cholesky decomposition as described in Section 2 to transform the stochastic differential equations. For comparison with Babbs and Nowman (1999), we also impose the following three additional restrictions: the second factor has zero impact on the term structure at approximately the five-year maturity and the third factor loading disappears at about two and twelve years. This gives a set of nine equations in the nine volatility parameters of (4), (5) and (6).

Figure 4 plots the factor loadings for the three-factor model. Whereas Babbs and Nowman found that their third factor loading had a negligible effect, we find all three factors have a significant impact on the yields of all maturities. We also find that the range of the impact of the factors **R**, **Y** and **X** on the yields is similar to that found by

Litterman and Scheinkman (1991) using principal component analysis on weekly US data with these three factors interpreted as *level*, *curvature* and *steepness* respectively.

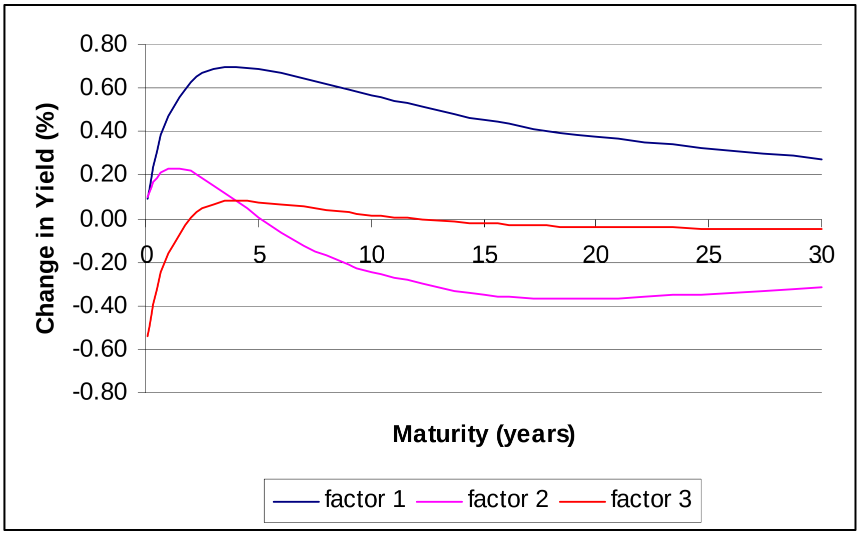

Figure 4. Factor loadings of the three-factor model

4. Simulation

One of the objectives of this paper is to propose a term structure model that is tractable in forward simulations through closed form yields given factors but can still capture the salient features of the yield curve.

Yield curve statistics

To evaluate our model initially, we performed an out-of-sample backtest over 2003. Using the historical 52 weekly data points for the yields in 2003, we calculated the mean level and the weekly standard deviation for each of the sixteen maturities. We then simulated forward from January 2003 to beginning January 2004 using the parameter estimates given in Table 1. In total 500 scenarios were generated and for each scenario the mean and standard deviation over time for the sixteen maturities was calculated. Averaging over all scenarios finally gives an average mean and standard deviation for the simulated yields.

Figure 5 plots the mean levels of the yields for both the historical and the simulated data and Figure 6 similarly plots the standard deviations. As can be observed from

Figure 5 the two sets of means closely match each other. Figure 6 shows that the simulated standard deviations slightly over-estimate the historical ones. However, yields were considerably less volatile in 2003 than over the 1997-2002 in-sample period (*cf.* Figure 2) which would explain this discrepancy.

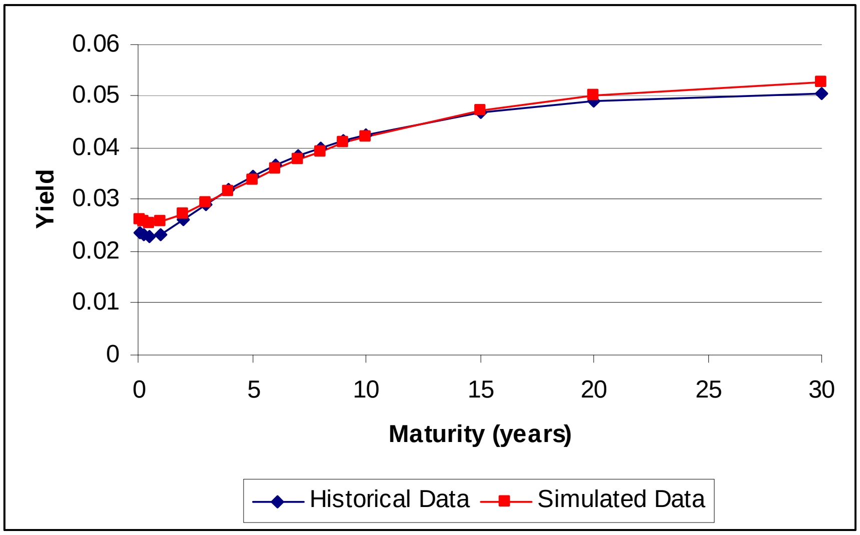

Figure 5. Mean level of yields over 2003 for historical and simulated data

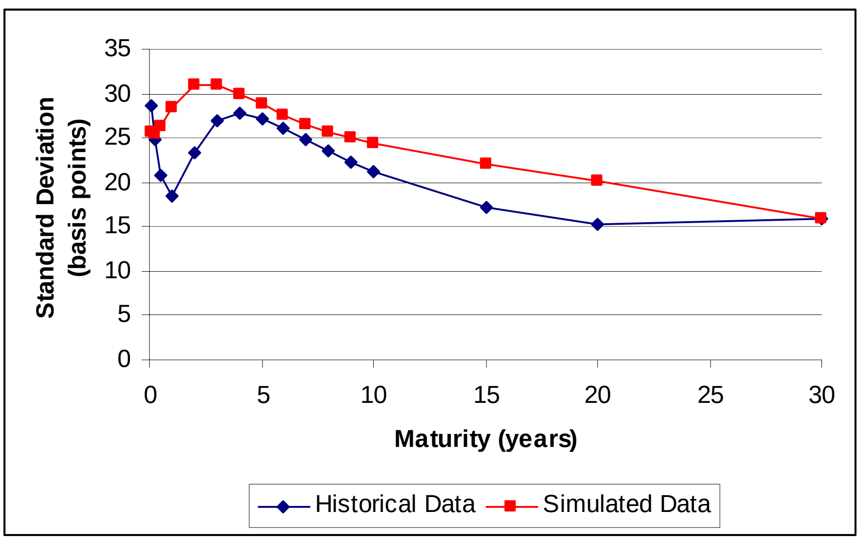

Figure 6. Weekly standard deviation of yields over 2003 for historical and simulated data

Yield curve dynamics

Another objective of this work was to develop a model that is able to simulate the various yield curve dynamics encountered in practice, e.g. steepening, flattening and inversion. Figures 7 and 8 show historical yields up to 2002 followed by simulated yields for two years to 2005 on specific yield curve scenarios selected from the 500 simulated[7]. Figure 7 demonstrates that the model can simulate yield curve steepening and flattening, while Figure 8 demonstrates that it can simulate yield curve inversion. Indeed, both visual and statistical analysis of the 500 simulated scenarios (not presented here in the interests of brevity) demonstrate that the model's simulated dynamic behavior is consistent with the historical behaviour of the yield curve over the out-of-sample period. This has been true for the many different applications with different time steps for which we have used it.

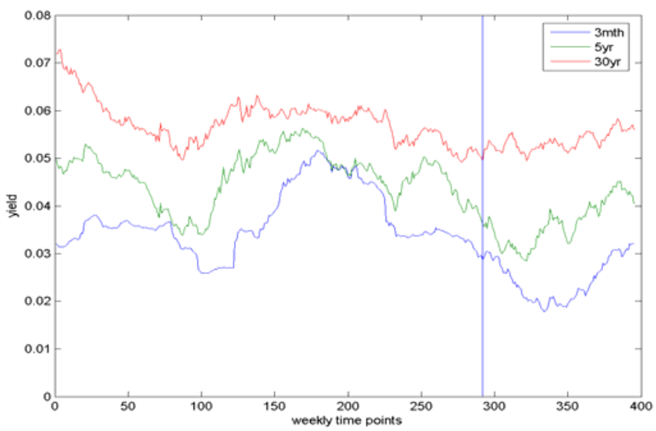

Figure 7. Forward simulation showing yield curve steepening and flattening

[7] Given the relatively low yield volatilities depicted in Figure 6 and the yield levels in Figure 5 we concluded that the probability of negative yields with our Gaussian model under the market measure is negligible. Using values from the data of Figures 5 and 6 these correspond to a minus 10 standard deviation event. Our decision is borne out by the representative paths in Figures 7 and 8 and in fact none of the 500 scenarios simulated produced negative yields over the out-of-sample period. However, see Abu-Mostafa (2001) for a technique for reducing this probability over longer simulation horizons.

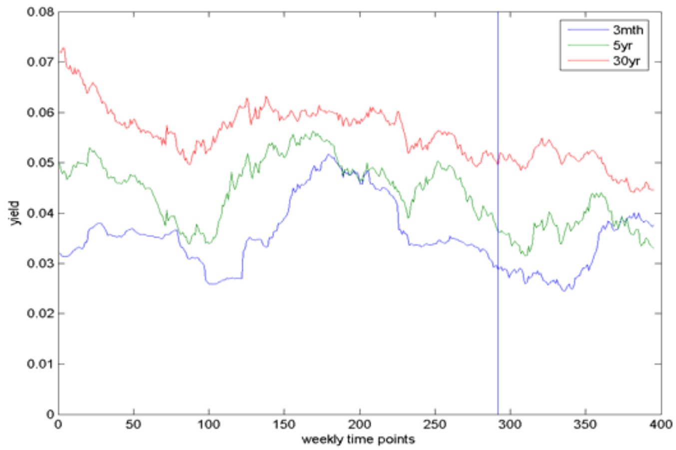

Figure 8. Forward simulation showing yield curve inversion

Application of the yield curve model to the risk management of portfolios of guaranteed investment products involving simulations with a monthly time step may be found in Dempster *et al*. (2006, 2007, 2009).

5. Pricing Consol Bonds

In 2005 a number of European banks[8] issued floating rate callable *consol* bonds as a means of raising Tier 1 regulatory capital. In the absence of the exercise of the call option by the bank at any time after a specified number of fixed interest payments, these bonds are *perpetual*, i.e. they have an infinite maturity, and their holders have purchased an indefinite income stream in exchange for their capital. After the period specified the fixed rate payable by the banks to the holder on the nominal face value of the bond is converted to a floating rate which is a multiple of the CMS-spread, usually the difference between the 10 and 2 year *constant maturity swap* (CMS) rates, on the interest payment fixing date. In addition, these bonds' payments normally have a cap and a floor, possibly in terms of another floating money market rate such as 3-month Euribor. In the *worst* case, when the swap curve is flat (see Figure 10) and the spread negligible, the bond holder will receive the floor rate.

[8] For example, in 2005 Deutsche Bank issued a € 900 M tranche of bonds at par to *face value* or *nominal*. This revives an instrument that has not been in favour since the Russian Revolution, when Tsar Nicholas' consols became worthless, although UK consols initiated in the 18th century are still in existence (with reduced fixed coupon). There is little current literature on their pricing when coupon rates are floating.

The details of *over-the-counter* (OTC) contracts are important and we present here a representative example of a callable CMS-spread consol hybrid product used to generate Tier 1 capital by the issuing bank.

Example

Nominal (face) value: € 1.5 M

Commencement date of contract: 28th January 2005

Maturity: perpetual (unless called at 5 years or after)

The bank pays a fixed rate of 6 % per annum in arrears for 5 years. The interval between payments on the 28th January each year is 1 year. At these dates the annual floating rate payments in arrears are calculated as

$$4(CMS10_i - CMS2_i) \quad i = 6,7,\ldots,$$

where CMS10 is the 10 year swap rate (base rate 10) and CMS2 is the 2 year swap rate (base rate 2) and $(CMS10_i - CMS2_i)$ is referred to as the *spread.* This floating rate coupon is capped at 10 % per annum and floored at 3.5 % per annum. As the spread decreases to 0 the bondholder receives only the floor rate of 3.5 % per annum.

The expected value of this bond for the purchaser (i.e. investor) is embedded in their belief about future movements of the swap curve over time. Historical swap curves illustrate that the expected and realized market rates may differ significantly. Figure 9 shows the movements of swap curves in 2005 and 2006.

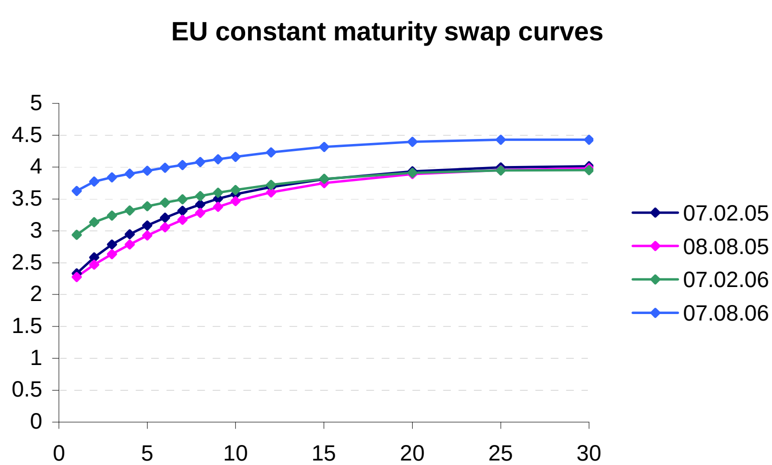

Figure 9. Illustrative swap curve movements

It is obvious that interest rate and, more generally, swap curve forecasting rests at the centre of CMS-spread instrument pricing for both seller and purchaser. Analysis of the risk of the proposed bond depends upon the counterparties abilities to model and simulate forward the yield and swap curves at the payment dates under the market measure. The broad movements of the spreads between 10 and 2 year maturities for the yield curve (for zero-coupon Treasury bonds) and the swap curve are similar. However, the swap curve spread has extra volatility due to the market's changing views on general counterparty creditworthiness – essentially AA credit rating spread volatility.

Depending upon market conditions the credit spreads of consols similar to our example appear to be associated with credit ratings which vary between A^- and BB^- in Standard and Poor (S & P) terms. However some of this market spread may be due to the mainly individual investors who bought and are currently trading these perpetual securities evaluating them as relatively long but *finite* lived bonds. For example, currently valuing the perpetual bond as having a 25 year maturity *without* credit risk produces a discount to face value of about 30 %, near the current trading range.

The holder of a CMS-spread consol bond is in effect giving the issuer a levered call option on the flattening of the yield curve – i.e. the *decrease* of the spread – which normally follows sharp rises in short term rates. Global macroeconomic conditions in the late 2004 to early 2005 period in which these contracts were issued clearly indicated sharply increasing short rates, a process that had already begun in the US at the time and followed in the EU only shortly thereafter. Moreover this was recently the situation due to credit market turmoil with short rates extremely high. However, the situation from January 2011 forward is naturally unclear at this point. There exists considerable asymmetry of information between issuer and bondholder and the latter may genuinely be convinced that "the yield curve will not become significantly flatter" in the long run to yield around 6 % in perpetuity.

Calibration

The data used to calibrate the model consists of freely available *daily* euro 3 and 6 month LIBOR and Euro swap data with the same maturities as in Section 3 from the start of 1999 to the end of 2007, a total of 2,133 observations[9]. Although the historical EU data often used by banks goes back to 1992, this data has had to be constructed prior to the introduction of the euro in 1999. In any event, missing the sharp short rate rises of the early 1990's in our data will tend to make our valuation estimates conservative. We again interpolate the swap curve linearly to obtain swap rates at all maturities, then use the 3 and 6 month EU LIBOR rates and the swap curve to recursively back out a risk free zero-coupon bond yield curve from the basic par swap pricing equation for each day. This derived data is the input data for estimation of the parameters of our 3-factor yield curve model (which we shall do at the date of bond issue and a more recent date). From this we can compute the yield curve based on the posterior mean for the three factors **R**, **X** and **Y** at historical dates in our data and compare this to the actual yield curve deduced from the (linearly interpolated) historical swap curve on that day. This is shown below in Figure 10 for a representative date, 28 July 2003, after calibration to the data up to consol bond inception at 28 January 2005[10].

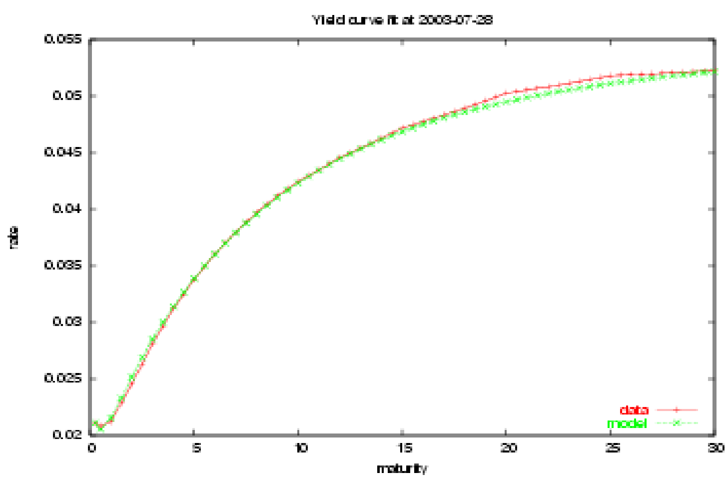

Figure 10. Yield curve fit for 28 July 2003

[9] We use daily data for consol bond valuation to conform to market practice by issuers who value fixed income instruments incorporating yield curve data on (or just before) the day of sale. Our example here is representative of a number of consol bonds we have valued initially on different dates in the period 2004 to 2006.

[10] Note that these fits on representative days do not always accurately capture the long end of the yield curve, which might require a fourth factor. They are however acceptably accurate up to 10 year maturity and in any event generally err on the conservative side, by producing lower discount rates.

For such calibrations on daily data the parameter estimates for the model have similar characteristics to those given for weekly data in Tables 1 and 2 in Section 3. Although long run means and mean reversion speeds are quite stable, market prices of risk and volatilities vary with calibration end date. Moreover, as we shall see below, the resulting consol bond Monte Carlo valuations vary considerably with the initial yield, and corresponding swap, curves – actually their model approximations at the calibration end date – from which forward simulation paths for valuation on that date begin.

Ignoring for the moment the ability of the bank to call (cancel) the bond, to find its fair price in the absence of credit risk we simulate the swap curve forward under the risk adjusted probabilities (i.e. with factor market prices of risk set to 0) using our interest rate model.

We compute the floating payment on each simulated scenario at each payment date and average across scenarios the total of the discounted payments along each random scenario. This is the standard Monte Carlo pricing methodology for European-style financial instruments. We used 50,000 paths for pricing the contract.

Clearly the present value of an infinite stream of payments from this consol cannot be obtained mathematically and must be valued numerically over a finite horizon. We chose this horizon by the criterion of the maturity at which the present value at inception of the remaining coupon payments thereafter (assumed to be at the 10 % cap and discounted conservatively at 2.5 % per annum) is less than one percent of face value, which occurred at 241 years.

With the right to cancel in place, the fair value is given by the expected discounted value of the sum of the coupon payments with the risk neutral probabilities under the assumption that the bank uses an optimal call strategy. Since determining the exact optimal cancellation rule is computationally difficult, we use a sub-optimal cancellation rule derived using the popular method of Andersen (1999). Due to the fact that only the bank has the right to cancel, the sub-optimality of our cancellation strategy may lead to an *over estimation* of the value of contract from the viewpoint of the bond holder.

In brief, Andersen's method relies on a *score* function $s_t(r,x,y)$ which should be low if cancellation is likely to be correct and seeks a cancellation rule of the form: cancel if $s_t < s_t^*$. The exercise thresholds s_t^* are determined recursively based on a *separate* set of random paths for (**R**, **X**, **Y**). We used 10,000 paths to estimate the optimal cancellation thresholds. Andersen proposed a simple method for determining good values for s_t^*. For our calculations we take s_t^* to be the discounted value of all the remaining swap payouts under the assumption that (**R**, **X**, **Y**) evolves according to its expected path.

We further improve the cancellation strategy as follows. Before evaluating the score function we compute an accurate approximation to the expected value of the next payout (since this is the payment committed to by opting not to cancel the contract at this time) by linearizing the expression for the spread (CMS10 - CMS2) at the end of the next period as a function of (R,X,Y). This leads to an integral involving two correlated Gaussian random variables which can be evaluated in closed form. If the expected next net return to the bank on the face value relative to the coupon paid to the bondholder is positive it cannot be correct to call the bond and it is better to wait for at least one more coupon payment.

To handle the credit spread due to the creditworthiness of these subordinated consol bonds we assume a *constant* per annum default rate appropriate to the credit class. This class could possibly be defined from market conditions (current yield curve) and similar instruments trading at different discounts due to different terms (nominal rates over 3-month EU LIBOR or EURIBOR). We might therefore have chosen to use the corresponding default rate at a constant 2.3 % per annum which represents the margin over EURIBOR of the interest rate paid for similar consol bonds issued at par but on more favourable terms – and currently trading near par – by other institutions than our issuing bank in the same period. This corresponds to a Standard and Poor (S&P) BB^- credit rating historical default rate. However, this default rate in inconsistent with the A credit rating initially assigned to this bond and below we shall actually approximate the credit discount *implied* by the market for a 241 year maturity bond.

NPV value at risk

Value at risk (VaR) can be computed at any point in time for the bondholder from a *simulated distribution* of the *present value* (PV) of all future (*net*) payments of the deal treating the initial face value payment as a sunk cost to the bond holder. We compute value at risk for the deal at inception and about three years later using exactly the same Monte Carlo methodology as that used for pricing, except that the market probabilities, involving estimates of the constant market prices of risk for the three factors, are used. We then compute normalized histograms of deal present values and find the 99 % VaR level (relative to 1 representing par or face value) for both the issuing bank and the bondholder. Since the factor market prices of risk are in fact *processes*, the standard deviations of their *constant* estimates are high relative to those of other parameters. Moreover, the estimates of the deal present value distributions are sensitive to the estimated value of the market price of risk for the short rate used to discount future payments (Cairns, 2004). We have attempted to overcome this effect in the estimation procedure by penalizing deviations from the (estimated) long run (asymptotic) short rate, which can be obtained in closed form as a function of the model parameters and is estimated from the historical three month EURIBOR and euro LIBOR rates.

Cash flow analysis

The swap rates for CMS2 and CMS10 (Figure 11) evolved significantly over the two years 2005, 2006 which moved the spread beyond (i.e. to 0.006 on 21.12.06) its historical minimum value up to 2004 of 0.3250 %. This has had a dramatic effect on the forecast floating rate coupon payments to the bond holder after the 5 year 6 % fixed rate period (see Figure 13).

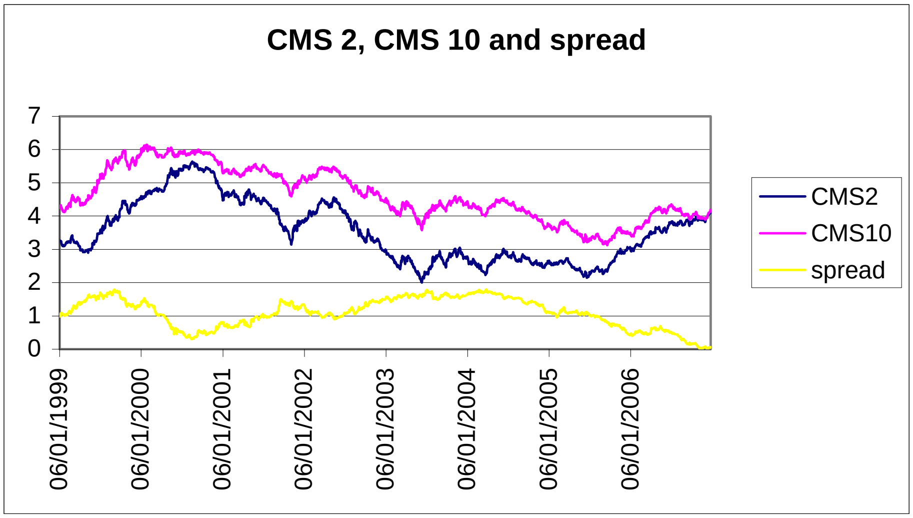

Figure 11. Base rate and spread evolution

Figure 12 presents the results of simulation of the coupon payments with the mean and range given by one standard deviation which seems beneficial for the bond holder at inception on 28th January 2005 assuming *no bond call or default* by the bank. Our model predicts that the evolution of this distribution of net payments from inception is not symmetrical about its mean, which falls slightly over the life of the bond. Note that the first five points in Figure 12 represent the first five fixed annual payments to the bond holder.

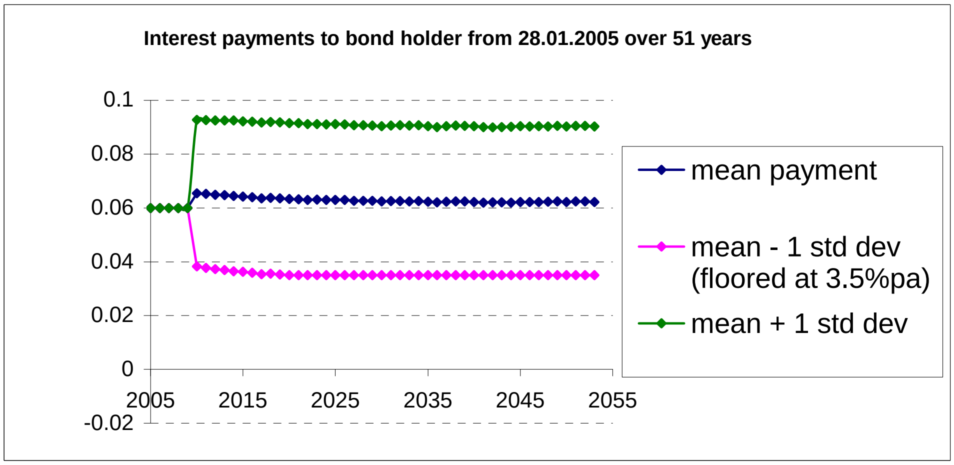

Figure 12. View of forward coupon payment distribution on 28.1.05

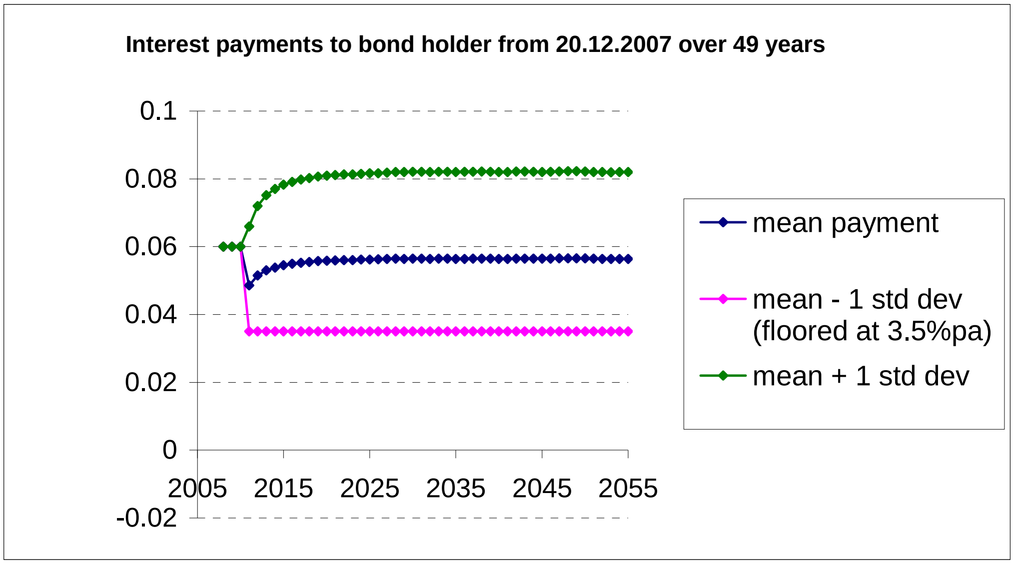

Figure 13. View of forward coupon payment distribution on 20.12.07

Figure 13 shows the simulation of the coupon payments for the bond, assuming no call or default by the bank, from the model calibrated on data to 20th December 2007 – a year when significant changes in the spread occurred (see Figure 11). In fact the long run stationary distribution of the spread under the market probabilities is 1.03 % per annum with corresponding 3-month short rate of 3.11 % per annum. This spread is similar to the 1.47 % historical average spread from 1995 to 2005 but lower due to more recent history in which short rates were rising and long rates were depressed by global liquidity prior to the credit crisis. Under the risk discounted (pricing) probabilities (risk neutral measure) the spread becomes -3.3 basis points with short rate 8.42 % per annum. Market moves over the two years from inception have been in a direction which has made the non-credit-risk adjusted CMS-spread bond deal costly in the outcome for the bond holder at 20th December 2007 at now less than 6 % per annum in expectation.

Deal valuations and values at risk

Market price of bond at inception (8.1.05)	86.05 % of face value or **€ 1.291 M**

Six standard deviation pricing uncertainty (99.7 % confidence interval):	85.57 % to 86.53 % of nominal or € 1.284 M to € 1.298 M

The **call option** at inception is **always used** prospectively.

Value at risk at inception to investor at the 99 % level:	46.27 % of face value recovered, i.e. 53.73 % or **€ 806 k lost**
Market price of bond (20.12.07) after two 6 % coupon payments	80.06 % of face value or **€ 1.207 M**
Six standard deviation pricing uncertainty (99.7 % confidence interval):	79.67 % to 80.44 % of face value or € 1.195 M to € 1.206 M
Value at risk on 20.12.07 to investor at 99 % level:	48.97 % of face value recovered, i.e. 51.03 % or **€ 765 k lost**

A number of observations are immediate. First, the deal was not initially fairly priced at par. Approximately 14 % of face value, or € 210 k was collected up front from the bond holder by the bank. The initial fair price of 86 % of face value is a higher than current market prices for these bonds which are currently in the range 60 to 65 % – perhaps due to higher credit risk or the individual investor finite horizon effect discussed above.

Secondly, in the absence of default, the bank's call (cancellation) option is optimally *always used*. The call date varies by scenario from 5 to 224 years from inception with an average of about 18 years.

Thirdly, the 99 % VaR to the bondholder involve considerable losses and depend critically on market conditions. Note however that 30% of the face value is recovered in the five fixed payments before discounting. The situation is illustrated graphically in Figures 14 and 15 which give the distributions of present value (PV) of future payments as a proportion of nominal (1 represents face value). These figures show the asymmetry in these distributions with a long thin upside tail in favour of the bond holder but also a significantly *probable downside* in favour of bank. From inception at 28.1.05 to 20.12.07 the expected present value has been reduced significantly from 1.154 (115.4 % of face value) to 1.055 (105.5 %) although the more appropriate median present value has been reduced somewhat less.

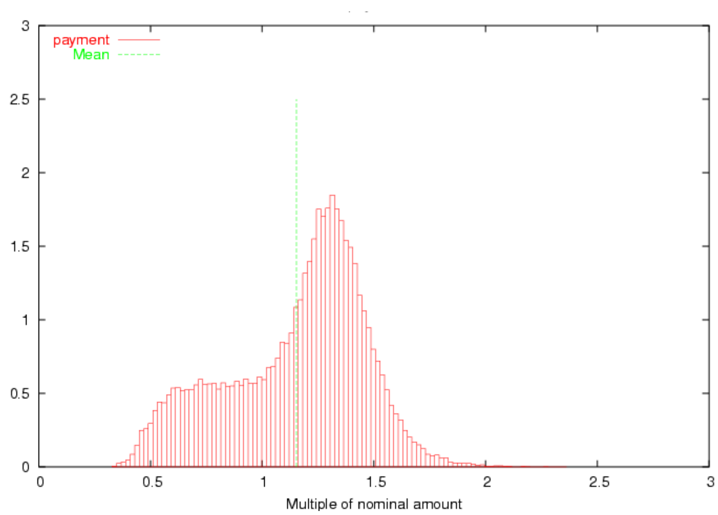

Figure 14. Distribution of total discounted payments to investor in multiple of face value at 28.1.05

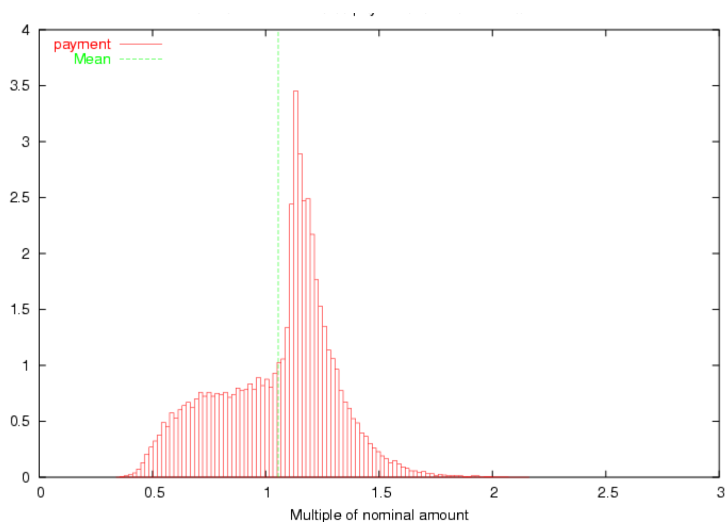

Figure 15. Distribution of total discounted payments to investor at 20.12.07

Comparison with risk free bonds

By comparison with the 115.4 % PV of future payments at inception of the actual contract on 28th January 2005 (Figure 14), the corresponding figure for an 18 year maturity 6 % fixed coupon risk free bond (with no call option) is 120.6 %. The 5.2 %

reduction represents the balance between the average effects of the call option and the potential for coupon payments near the floor (reducing) and the potential for higher coupon payments of up to 10 % over periods when the bank optimally calls the perpetual bond later than the average 18 years (enhancing). Nearly three years later, on 20th December 2007 (Figure 15) when the PV of future payments is only 105.5%, the corresponding reduction relative to a 15 year maturity 6% risk free bond exceeds 15%. At this date the likelihood of the bond holder not even recovering the original investment has risen to nearly 50%.

Credit risk analysis

Finally, let us consider the effects of credit risk on the current market price of this consol bond. Taking account of credit risk for such a perpetual bond from a necessarily (short) finite amount of historical default data is fraught with error. We have chosen to use the maturity of 241 years of our finite maturity approximation to obtain at least a plausible value for the credit risk discount. The difference between our market risk valuation of 80% of face value on 20th December 2007 and the market value of the bond on that day of 60% allows 20% of face value as credit risk discount and / or behavioural finite horizon and tax effects. Assuming that this figure is essentially due to *pure credit risk* implies that the market on that day was assuming approximately a default discount rate of 8.3 bp per annum[11], which would lead to a credit risk discount over 241 years of 20.1%.

The only rational explanation for the purchase of this representative consol is that bond holders believed that their coupon payments would genuinely not be significantly reduced from the first five at 6 %. As we have seen (in Figure 12) this would have been a reasonable expectation based on history at inception, but actual market outcomes – possibly revealed to the issuing bank in the forward economic and market views at inception – have further moved strongly against the bond holders (see Figure 13). The result is that an investor is left holding an illiquid credit risky perpetual bond which is currently trading at a 30 % to 40 % loss on their investment

[11] This corresponds to the historical *four year* S&P cumulative default rate for the bond's A rating which suggests that the market was optimistic regarding the bank's possible default on the contract over possibly nearly two and a half centuries.

with the prospect in two years time of possibly receiving only 3.5 % annual coupons due to CMS-spreads of only a few basis points.

Obviously these credit risky structural floating rate consol bonds provide investors with returns far inferior to risk-free fixed rate bonds of comparable expected maturities issued in the same period. However the complexity of these instruments, which require sophisticated Monte Carlo analysis to price, has by and large been ignored by investors. Indeed, initially investors appear to have treated these securities naively, and sub-optimally, as short maturity risk-free fixed rate bonds which would be called by banks soon after all their initial fixed rate payments were made.

6. Conclusion

The objective of this paper is to specify a model that captures the salient features of the whole term structure, rather than one that just focuses on the short-term interest rate. It also has to be tractable in order to form a basis for asset pricing applications and forward simulations for asset liability management. To this end, we consider a Gaussian three-factor continuous-time model within the affine class with a closed-form solution for bond prices.

For our empirical analysis, the model is expressed in a state-space formulation which allows us to take into account both the cross-sectional and time-series information contained in the term structure data and to use the Kalman filter and numerical likelihood maximization recursively to estimate the parameters.

The model explains the cross-section of interest rates well with reasonably small yield errors. We also show that in forward simulations this model gives rise to a wide and realistic range of future interest rate scenarios, as shown by both a backtest and simulation results involving flattening / steepening / inversion of the yield curve.

We apply the model to pricing perpetual callable consol bonds with structured floating coupon payments, based on the 10-2 year constant maturity swap spread, using forward simulations over a 241 year horizon with a daily time step. As a result we find these credit risky floating rate consol bonds issued by banks to raise Tier 1 capital in the 2004-2005 period to be initially mispriced and with lower expected yields than comparable finite horizon sovereign fixed coupon bonds.

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